Math 246C Lecture 4 Notes

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1 Lifting of Homotopic Curves and Existence of Lifts

1.1 Lifting of homotopic curves

Last time we introduced the idea of a covering map $p : Y \to X$. It has the following path lifting property:

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & X \end{tikzcd}$$

Remark 1.1. If X is path-connected (ok for Riemann surfaces), then $p : Y \to X$ is surjective: Let $x_0, x_1 \in X$, and let γ be a path joining x_0, x_1. Then for any $y \in p^{-1}(x_0)$, there eis a unique lift $\tilde{\gamma}[0,1] \to Y$ such that $\tilde{\gamma}(0) = y$ and $\tilde{\gamma}(1) \in p^1(x_1)$. This gives rise to a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$. Moreover, the cardinality of $p^{-1}(x)$ is constant.

Theorem 1.1 (lifting of homotopy curves[1]). *Let X, Y be Hausdorff, and let $p : Y \to X$ be a local homeomorphism. Let $a, b \in X$, and let $\gamma_0, \gamma_1 : [0,1] \to X$ be paths joining a to b that are homotopic. There exists a continuous deformation $H(t,s) : [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma_0(t)$, $H(t,1) = \gamma_1(t)$, $H(0,s) = a$, and $H(1,s) = b$.*

Let $\gamma_s(t) = H(t,s)$. Let $a_1 \in p^{-1}(a)$, and assume that each γ_s has a lift $\tilde{\gamma}_s$ to y such that $\tilde{\gamma}_s(0) = a_1$. Then $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are homotopic and have the same endpoint.[2]

Proof. Set $\tilde{H}(t,s) = \tilde{\gamma}_s(t)$ for $0 \leq t, s \leq 1$. Let us show first that $\tilde{H}$ is continuous. We claim that there exists some $\varepsilon_0 > 0$ such that $\tilde{H}(t,s)$ is continuous on $[0, \varepsilon_0] \times [0,1]$. We have $H(\{0\} \times [0,1]) = \{a\}$. Let $V \subseteq Y, U \subseteq X$ be neighborhoods of a_1 and a such that $p|_V : V \to U$ is a homeomorphism. By compactness of $[0,1]$ and continuity of H, there exists $\varepsilon_0 > 0$ such that $H([0, \varepsilon_0] \times [0,1]) \subseteq U$. Let $\varphi = (p|_V)^{-1} : U \to V$. The curve

[1] This theorem is sometimes called the abstract monodromy theorem.

[2] Professor Hitrik says "some theorems may not be meant to be discussed in public." After seeing the proof of this, you may agree.

$[0, \varepsilon_0] \ni t \mapsto \varphi(\gamma_s(t))$ is a lift of γ_s on $[0, \varepsilon_0]$, $0 \le s \le 1$, and by the uniqueness of lifts, $\varphi(\gamma_s(t)) = \tilde{\gamma}_s(t) = \tilde{H}(t, s)$ on $0 \le t \le \varepsilon_0$. We get the claim.

We now claim that $\tilde{H}$ is continuous on $[0,1] \times [0,1]$. Assume that the claim fails, and let (t_0, σ) be a point of discontinuity of $\tilde{H}$. Let $\tau = \inf\{t : \tilde{H} \text{ is not continuous at } (t, \sigma)\}$. Then $0 < \varepsilon \le \tau$. Let $x = H(\tau, \sigma)$ and $y = \tilde{(}\tau, \sigma)$; that is, $t = \tilde{\gamma}_\sigma(t)$, so $y \in p^{-1}(x)$. Let V, U be neighborhoods of y and x such that $p|_V : V \to U$ is a homeomorphism, and let $\varphi = (p|_V)^{-1}$. By continuity of H, there exists $\varepsilon > 0$ such that $H(I_\varepsilon(\tau), I_\varepsilon(\sigma)) \subseteq U$, where $I_\varepsilon(\tau)$ is a neighborhood of τ and $I_\varepsilon(\sigma)$ is a neighborhood of σ. In particular, $\gamma_\sigma(I_\varepsilon(\tau)) \subseteq U$. We can also assume that $\tilde{\gamma}_\sigma(I_\varepsilon(\tau)) \subseteq V$. We get $\tilde{\gamma}_\sigma(t) = \varphi(\gamma_\sigma(t))$ for $t \in I_\varepsilon(\tau)$. Let $t_1 \in I_\varepsilon(\tau)$ with $t_1 < \tau$. Then $\tilde{H}$ is continuous at (t_1, σ), so there is a neighborhood $I_\delta(\sigma)$ of σ with $\delta \le \varepsilon$ such that $\tilde{H}(t_1, s) \in V$ for $s \in I_\delta(\sigma)$. Now $t \mapsto \tilde{\gamma}_s(t)$ and $t \mapsto \varphi(\gamma_s(t))$ for $t \in I_\varepsilon(\tau)$ are both lifts of $\gamma_s(t)$, and by the uniqueness of lifts, $\tilde{\gamma}_s(t) = \varphi(\gamma_s(t))$. In particular, $\tilde{H}$ is continuous in a neighborhood of (τ, σ), which contradicts the definition of τ. We get that $\tilde{H}$ is continuous on $[0,1] \times [0,1]$.

We also need to check that $s \mapsto \gamma_s(1)$ is constant. This is continuous and lifts the constant path $s \mapsto b$. By the uniqueness of lifts, $\tilde{\gamma}_s(1) = \tilde{\gamma}_0(1) \in p^{-1}(b)$. $\square$

1.2 Existence of lifts

Theorem 1.2 (existence of lifts)**.** *Let X, Y be Hausdorff spaces, and let $p : Y \to X$ be a covering map. Let Z be a Riemann surface which is simply connected, and let $f : Z \to X$ be continuous. For any $x_0 \in Z$ and $y_0 \in Y$ such that $f(z_0) = p(y_0)$, there is a unique lift $\tilde{f} : Z \to Y$ such that $\tilde{f}(z_0) = y_0$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

We will prove this next time. First, here are examples.

Example 1.1. Let $Y = \mathbb{C}$ and $X = \mathbb{C} \setminus \{0\}$. Then $p(z) = e^z$ is a covering map. If $f \in \mathrm{Hol}(Z)$ is nonvanishing, then there exists a holomorphic lift $\tilde{f}$ such that $e^{\tilde{f}} = f$.

Example 1.2 (Picard's little theorem)**.** Let $f \in \mathrm{Hol}(\mathbb{C})$ and $0, 1 \notin f(\mathbb{C})$. Then $f :$ $\mathbb{C} \setminus \{0,1\}$. We shall show that the disc D covers $\mathbb{C} \setminus \{0,1\}$:

$$\begin{tikzcd} & D \arrow[d, "p"] \\ \mathbb{C} \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & \mathbb{C} \setminus \{0,1\} \end{tikzcd}$$

Then $\tilde{f} : \mathbb{C} \to D$ is constant, as it is bounded and entire. So f is constant.